

# Billiards and the Five Distance Theorem II

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## 1 Introduction

This paper is a complement to the author earlier paper [F]. We consider a billiard table rectangle with perimeter of length 1. If a billiard ball is sent out from position  $F(1)$  at the angle of  $\pi/4$ , then the ball will rebound against the sides of the rectangle consecutively in points  $F(2), F(3), \dots$ . Let  $n \geq 5$  and  $\Phi = \{F(j) : 1 \leq j \leq n\}$  be the set of different points. An open connected subset of the perimeter of the billiard rectangle with different endpoints from the set  $\Phi$  is called *segment*. A segment with endpoints  $F(k), F(l)$ ,  $1 \leq k, l \leq n$ , is called *even* (or *odd*), and has *weight*  $|k - l|$  (or  $k + l$ ) if  $k, l$  are of the same (or different) parity. *Length* of a segment is a distance along the perimeter between its endpoints. A segment with the length less than  $\frac{1}{2}$  is called *short*. A segment is called *elementary* if there are no points of the set  $\Phi$  between its endpoints. Suppose  $\emptyset \neq V \subseteq \{F(1), F(n)\}$ . A segment  $I$  is *associated* with  $V$  if  $I$  is an elementary segment incident with an element of  $V$  or  $I \cap \Phi$  is nonempty set contained in  $V$ . Since  $n \geq 5$ , any segment associated with  $\{F(1), F(n)\}$  is determined by its endpoints uniquely. By  $(F(k), F(l))$  (respectively  $(F(k), F(l))_e$ ) we denote the short segment (respectively the elementary segment) incident with  $F(k)$  and  $F(l)$ . Note that  $(F(k), F(l))$  and  $(F(l), F(k))$  are the same segments.

Let  $\omega_1 < \omega_2$  be odd weights and  $\omega_0$  be an even weight of segments associated with  $\{F(1)\}$ , and let  $\omega_3 < \omega_4$  be other odd weights of segments associated with  $\{F(1), F(n)\}$  (see Corollary 2.2). Let  $a_i$  be the length of the segment with the weight  $\omega_i$ , and suppose that  $A_i$  is the set of all elementary segments with weight  $\omega_i$ ,  $i = 0, \dots, 4$ . The author [F] have proved that the weights of elementary segments have at most five different values  $\omega_0, \dots, \omega_4$ . Hence,

$$|A_0| + \dots + |A_4| = n.$$

If  $a_2 - \epsilon a_1 = a_3 - \delta a_4 = a_0$ , for some  $\epsilon, \delta \in \{-1, 1\}$ , then the following equations are satisfied

$$|A_2| + \epsilon |A_1| = |A_3| + \delta |A_4| = \frac{1}{2} \omega_0.$$

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Moreover, elementary segments with equal weights have equal lengths. In this paper we prove the following equalities (see Theorem 2.2):

$$|A_1| = \frac{\omega_1 - 1}{2}, \text{ and } |A_4| = n - \frac{\omega_4 - 1}{2}.$$

Notice that we can consider a general case, when the initial angle of the ball's motion is not  $\pi/4$ . By a linear transformation we can change the billiard table rectangle, so that the general case is transformed to the  $\pi/4$  case (except the result that elementary segments with equal weights have equal lengths).

## 2 The main result

Recall the following results of the paper [F].

**Remark 2.1** [F, Remark 2.2] *An odd elementary segment which is not short is of the form  $(F(1), F(2))_e$  or  $(F(n-1), F(n))_e$ .*

**Theorem 2.1** [F, Theorem 2.2] *If  $(F(k), F(1))_e, (F(1), F(n))_e, 1 < k < n$ , are elementary segments incident with  $F(1)$ , then  $k$  is even.*

**Corollary 2.1** [F, Corollary 2.1] *If  $(F(1), F(n))_e, (F(n), F(l))_e, 1 < l < n$  are elementary segments incident with  $F(n)$ , then  $l, n$  are of different parity.*

**Lemma 2.1** [F, Lemma 2.3(1)] *Let  $k+1 < l$  and  $k, l$  be of different parity. Then we have: a short segment  $(F(k), F(l))$  is elementary, or is associated with  $\{F(1), F(n)\}$  if and only if the short segment  $(F(k+1), F(l-1))$  is elementary.*

**Corollary 2.2** [F, Corollary 2.2] *There exist exactly two odd segments and one even segment associated with  $\{F(1)\}$ , and exactly four odd segments associated with  $\{F(1), F(n)\}$ . The weights of these four odd segments are different.*

Now we present the main result of this paper.

**Theorem 2.2** *Let  $\omega_1 < \omega_2$  be odd weights and  $\omega_0$  be an even weight of segments associated with  $\{F(1)\}$ , and let  $\omega_3 < \omega_4$  be other odd weights of segments associated with  $\{F(1), F(n)\}$ . If  $A_1$  ( $A_4$ ) is the set of all elementary segments with weight  $\omega_1$  ( $\omega_4$ , respectively), then*

$$|A_1| = \frac{\omega_1 - 1}{2} \quad \text{and} \quad |A_4| = n - \frac{\omega_4 - 1}{2}.$$

**Proof.** Suppose  $(F(k), F(1))_e, (F(1), F(l))_e, k < l$ , are elementary segments incident with  $F(1)$ , and  $(F(r), F(n))_e, (F(n), F(t))_e, r < t$ , are elementary segments incident with  $F(n)$ . By Theorem 2.1 and Corollary 2.1

$(F(n), F(t))_e$  are odd. If  $(F(k), F(1))_e$  (or  $(F(n), F(t))_e$ ) is short, then by Lemma 2.1,

$$|A_1| = \frac{k}{2} = \frac{\omega_1 - 1}{2}$$

(or

$$|A_4| = \frac{n - t + 1}{2} = \frac{2n - \omega_4 + 1}{2},$$

respectively). If  $(F(k), F(1))_e$  (or  $(F(n), F(t))_e$ ) is not short, then by Remark 2.1 and Lemma 2.1

$$k = 2 \quad \text{and} \quad |A_1| = 1 = \frac{\omega_1 - 1}{2}$$

(or

$$t = n - 1 \quad \text{and} \quad |A_4| = 1 = \frac{2n - \omega_4 + 1}{2}$$

respectively). □

## References

- [F] J. Florek, Billiards and the five distance theorem, *Acta Arith.* **139.3** (2009), 229–239.

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